CONVERSE THEOREMS OF SUMMABILITY FOR DIRICHLET'S SERIES*

BY

OTTO SZÁSZ

1. Let the Dirichlet series

$$F(t) = \sum_{\nu=1}^{\infty} c_\nu e^{-\lambda_\nu t}, \qquad 0 < \lambda_1 < \lambda_2 < \cdots, \ \lambda_n \to \infty, \tag{1}$$

be convergent for $t>0$. Let, in addition, the limit

$$\lim_{t\to+0} F(t) = s \tag{2}$$

exist. It is well known that this is certainly the case whenever the series $\sum c_\nu$ converges. But the converse, in general, is not true, as is shown for instance by the example $\lambda_\nu = \nu$, $c_\nu = (-1)^\nu$,

$$F(t) = \sum_1^\infty (-1)^\nu e^{-\nu t} = -e^{-t}(1+e^{-t})^{-1} \to -\tfrac{1}{2} \text{ as } t \to 0.$$

Thus we are led to the problem of finding additional conditions which together with the assumption (2) would assure the convergence of $\sum c_\nu$.

Such conditions are the following:

$$c_n = O\left(\frac{\lambda_n - \lambda_{n-1}}{\lambda_n}\right) \text{ as } n \to \infty.† \tag{3}$$

$$\limsup_{m\to\infty} \ \underset{\lambda_m(1+\delta)^{-1} \leqq \lambda_n \leqq \lambda_m(1+\delta)}{\text{maximum}} |s_n - s_m| = \psi(\delta) \to 0 \text{ as } \delta \to 0, \quad \frac{\lambda_{n+1}}{\lambda_n} \to 1, \tag{4a}$$

and

$$s_n = \sum_{\nu=1}^{n} c_\nu = O(1) \text{ as } n \to \infty \tag{4b}$$

[Landau, 6].

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† See Littlewood [7] in case $\lambda_{n+1}/\lambda_n \to 1$, Hardy and Littlewood [4] in case

$$\liminf_{n\to\infty} \frac{\lambda_{n+1}}{\lambda_n} > 1,$$

Ananda-Rau [1] for the rest. The numbers in brackets refer to the list at the end of this paper.

$$\limsup_{x\to\infty} \sum_{x\leqq\lambda_k}^{\leqq(1+\delta)x} |c_k| = \eta(\delta) \to 0 \text{ as } \delta \to 0, \quad \frac{\lambda_{n+1}}{\lambda_n} \to 1 \tag{5}$$

[Neder, 8].

$$\sum_{\nu=2}^{n} |c_\nu|^p \lambda_\nu^p (\lambda_\nu - \lambda_{\nu-1})^{1-p} = O(\lambda_n), \quad p > 1, \quad \frac{\lambda_{n+1}}{\lambda_n} \to 1 \tag{6}$$

[Szász, 9].

$$\liminf_{x\to\infty} \ \underset{x\leqq\lambda_\nu\leqq(1+\delta)x}{\text{minimum}} \sum c_\nu = \phi(\delta) \to \gamma \geqq 0 \text{ as } \delta \to 0 \tag{7}$$

[Szász, 10].

2. The proof of (5) can be reduced to the special case $\lambda_\nu = \nu$ without using the condition $\lambda_{n+1}/\lambda_n \to 1$. Thus condition $\lambda_{n+1}/\lambda_n \to 1$ can be omitted in (5). Indeed, let

$$\sum_{\nu-1<\lambda_k}^{\leqq\nu} c_k = b_\nu, \quad \sum_{\nu-1<\lambda_k}^{\leqq\nu} |c_k| = \beta_\nu \qquad (\nu = 1, 2, 3, \cdots).$$

Then for any $\delta > 0$

$$|b_\nu| \leqq \beta_\nu < \eta(\delta) + \delta, \text{ for } \nu > n(\delta),$$

so that

$$\lim_{\nu\to\infty} b_\nu = \lim_{\nu\to\infty} \beta_\nu = 0, \tag{8}$$

and the series

$$f(t) = \sum_{\nu=1}^{\infty} b_\nu e^{-\nu t}$$

converges for $t > 0$. Moreover

$$F(t) - |f(t)| = \left| \sum_{\nu=1}^{\infty} \sum_{\nu-1<\lambda_k}^{<\nu} c_k(e^{-\lambda_k t} - e^{-\nu t}) \right| \leqq \sum_{\nu=1}^{\infty} \sum_{\nu-1<\lambda_k}^{<\nu} |c_k| (e^{-(\nu-1)t} - e^{-\nu t}).$$

Thus by (8)

$$|F(t) - f(t)| \leqq (1 - e^{-t}) \sum_{\nu=1}^{\infty} \beta_\nu e^{-(\nu-1)t} \to 0 \text{ as } t \to 0,$$

and, by (2),

$$f(t) \to s \text{ as } t \to 0. \tag{2*}$$

Since

$$\sum_{x \leqq \nu}^{\leqq (1+\delta)x} | b_\nu | \leqq \sum_{x-1 \leqq \lambda_k}^{\leqq (1+\delta)x} | c_k |$$

and

$$(1 + \delta)x \leqq (1 + 2\delta)(x - 1) \text{ for } x \geqq \frac{1}{\delta} + 2,$$

$$\limsup_{x\to\infty} \sum_{x \leqq \nu}^{\leqq (1+\delta)x} | b_\nu | \leqq \eta(2\delta) \to 0 \text{ as } \delta \to 0.$$

Consequently $\sum b_\nu$ converges, which in view of (8) implies the convergence of $\sum c_\nu$.†

3. A similar reduction and generalization is possible for the case (4). Since the expression $e^{-\lambda_k t} - e^{-\nu t}$ is >0 and monotonely decreasing in k, we have, by (4a),

$$\left| \sum_{\nu-1<\lambda_k}^{<\nu} c_k(e^{-\lambda_k t} - e^{-\nu t}) \right| \leqq (e^{-(\nu-1)t} - e^{-\nu t}) \max_{x \leqq \nu} \left| \sum_{\nu-1<\lambda_k}^{\leqq x} c_k \right|$$
$$\leqq (1 - e^{-t})e^{-(\nu-1)t}o(1) \text{ as } \nu \to \infty.$$

Hence again

$$F(t) - f(t) \to 0, \; f(t) \to s \text{ as } t \to 0.$$

Now on putting $\sum_{\nu=1}^{n} b_\nu = B_n$ we have

$$B_n = \sum_{\nu=1}^{n} \sum_{\nu-1<\lambda_k}^{\leqq \nu} c_k = \sum_{0<\lambda_k}^{\leqq n} c_k, \qquad B_n - B_m = \sum_{n<\lambda_k}^{\leqq m} c_k, \; n < m,$$

and, by (4a),

$$\limsup_{m\to\infty} \operatorname*{maximum}_{m(1+\delta)^{-1} \leqq n \leqq m(1+\delta)} | B_n - B_m | = \psi(\delta) \to 0 \text{ as } \delta \to 0.$$

We conclude that $\sum b_\nu$ is convergent and from

$$\lim_{x\to\infty} \max_{x \leqq \nu} \left| \sum_{\nu-1<\lambda_k}^{\leqq x} c_k \right| = 0$$

the convergence of $\sum c_\nu$ follows immediately.

The results of Neder and Landau without the assumptions $\lambda_{n+1}/\lambda_n \to 1$ and (4b) can also be derived from (7), for

$$\operatorname*{minimum}_{x \leqq \lambda_\nu \leqq (1+\delta)x} \sum c_\nu \geqq - \operatorname*{maximum}_{x \leqq \lambda_\nu \leqq (1+\delta)x} \left| \sum c_\nu \right|.$$

Finally, in (6) also the restriction $\lambda_{n+1}/\lambda_n \to 1$ can be removed. During the

† For a similar argument see Ananda-Rau [3].

writing of this paper there appeared an interesting paper by Iyer,* where a proof of this generalization is given. In the present paper I give a proof which is somewhat simpler, and a further generalization.

4. In what follows it is simpler to use Laplace integrals. We intend to prove two auxiliary theorems which are of interest in themselves.

THEOREM 1. *Assume that*

$$F(t) = t\int_0^\infty A(u)e^{-ut}du \text{ converges for } t > 0, \tag{1'}$$

$$F(t) \to s \text{ as } t \to 0, \tag{2'}$$

$$v(x) \equiv xA(x) - \int_0^x A(u)du \geqq -Kx,\ x > 0, \tag{9}$$

where K is a positive constant. Then

$$\frac{1}{x}A_1(x) \equiv \frac{1}{x}\int_0^x A(u)du \to s \text{ as } x \to \infty. \tag{10}$$

For the proof we need three lemmas.

LEMMA 1. *Assumptions* (1′) *and* (2′) *imply that the integral*

$$F_1(t) = t\int_0^\infty \frac{1}{u}A_1(u)e^{-ut}du$$

converges for $t>0$ *and*

$$F_1(t) \to s \text{ as } t \to 0. \tag{11}$$

On integrating by parts in (1′) and using (2′) we have

$$F(t) = t^2\int_0^\infty A_1(u)e^{-ut}du \to s,\ t \to 0,$$

where the integral converges absolutely for $t>0$. Hence

$$\int_t^\infty x^{-2}F(x)dx = \int_0^\infty A_1(u)du\int_t^\infty e^{-ux}dx$$

$$= \int_0^\infty \frac{1}{u}A_1(u)e^{-ut}du = O\left(\frac{1}{t}\right),\ t \to 0.$$

Now

$$F(t) - F_1(t) = t\int_t^\infty [F(t) - F(x)]\frac{dx}{x^2} = t\int_t^{t^{1/2}} + t\int_{t^{1/2}}^\infty,\ t < 1,$$

* [5]. In the proof on p. 112 the author refers to his Theorem 4 in the statement of which the "O" is to be replaced by "o," as is seen from the proof.

whence

$$|F(t) - F_1(t)| \leqq \max_{t \leqq x \leqq t^{1/2}} |F(t) - F(x)| + t^{1/2}|F(t)| + O(t^{1/2}) \to 0,$$

and so, by (2′), $F_1(t) \to s$ as $t \to 0$.

LEMMA 2. *If* $a(x) \to 0$ *as* $x \to \infty$, *then*

$$t\int_0^\infty a(u)e^{-ut}du \to 0 \quad as \quad t \to 0. \tag{12}$$

This lemma is well known.

LEMMA 3. *If* $xb(x) \to 0$ *as* $x \to \infty$, *and*

$$\int_0^\infty b(u)e^{-ut}du \to B \quad as \quad t \to 0, \tag{13}$$

then the integral $\int_0^\infty b(u)du$ *exists as an improper integral and*

$$\int_0^\infty b(u)du = B. \tag{14}$$

We have

$$\int_0^x b(u)du - \int_0^\infty b(u)e^{-ut}du = \int_0^x b(u)(1 - e^{-ut})du - \int_x^\infty b(u)e^{-ut}du$$
$$= H_1 - H_2.$$

For $t = 1/x$, $x \to \infty$, it follows that

$$|H_1| \leqq \frac{1}{x}\int_0^x u|b(u)|du = o(1),$$

$$|H_2| = o\left(\int_x^\infty u^{-1}e^{-u/x}du\right) = o\left(\frac{1}{x}\int_x^\infty e^{-u/x}du\right) = o(1).$$

We now pass on to the proof of Theorem 1. From (2′) and (11) it follows that

$$t\int_0^\infty \frac{v(u)}{u} e^{-ut}du \to 0 \text{ as } t \to 0,$$

whence

$$t\int_0^\infty \left(\frac{v(u)}{u} + K\right) e^{-ut}du \to K \text{ as } t \to 0.$$

By a well known theorem, this and (9) yield

$$\int_0^x \left(\frac{v(u)}{u} + K\right) du \sim Kx \text{ as } x \to \infty,$$

or

$$\frac{1}{x}\,\Phi(x) \equiv \frac{1}{x}\int_0^x \frac{v(u)}{u}\,du \to 0 \text{ as } x \to \infty. \tag{15}$$

Furthermore,

$$v(x) = xA(x) - A_1(x) = x^2 \frac{d}{dx}\frac{A_1(x)}{x}.$$

On assuming, as we may without loss of generality, $A(x) = o(x)$ as $x \to 0$, we have

$$\frac{A_1(x)}{x} = \int_0^x \frac{v(u)}{u^2}\,du, \tag{10'}$$

and, on integrating by parts,

$$\int_0^x \frac{v(u)}{u^2}\,du = \frac{1}{x^2}\,v_1(x) + 2\int_0^x \frac{v_1(u)}{u^3}\,du; \qquad v_1(x) \equiv \int_0^x v(u)du.$$

Thus (11) becomes

$$F_1(t) = t\int_0^\infty \left(\frac{1}{u^2}\,v_1(u) + 2\int_0^u \frac{v_1(\tau)}{\tau^3}\,d\tau\right) e^{-ut}du \to s, \quad t \to 0. \tag{11'}$$

But on integrating by parts we have by (15)

$$v_1(x) = x\Phi(x) - \int_0^x \Phi(u)du = o(x^2) \text{ as } x \to \infty, \tag{16}$$

whence, by Lemma 2,

$$t\int_0^\infty u^{-2}v_1(u)e^{-ut}du \to 0 \text{ as } t \to 0.$$

Now from (11′) it follows that

$$2t\int_0^\infty \left(\int_0^u \tau^{-3}v_1(\tau)d\tau\right) e^{-ut}du \to s \text{ as } t \to 0,$$

or, on integrating by parts again,

$$2\int_0^\infty u^{-3}v_1(u)e^{-ut}du \to s \text{ as } t \to 0.$$

Lemma 3 shows that

$$2\int_0^\infty u^{-3}v_1(u)du = s,$$

whence, by another integration by parts and by (16),

$$\int_0^\infty u^{-2}v(u)du = s.$$

Combined with (10′) this proves Theorem 1.

5. In order to apply this result to the series (1) put

$$\lambda_0 = 0,\ s_0 = 0,$$
$$A(x) = s_n \text{ for } \lambda_n \leqq x < \lambda_{n+1} \qquad (n = 0, 1, 2, \cdots).$$

A summation by parts yields $s_n = o(e^{\lambda_n t})$, $t>0$, and

$$F(t) = \sum_{\nu=1}^{\infty} s_\nu(e^{-\lambda_\nu t} - e^{-\lambda_{\nu+1} t}) = t\int_0^\infty A(u)e^{-ut}du,\ t > 0.$$

Furthermore, for $\lambda_n \leqq x < \lambda_{n+1}$,

$$\int_0^x A(u)du = \sum_{\nu=1}^{n} (\lambda_\nu - \lambda_{\nu-1})s_{\nu-1} + (x - \lambda_n)s_n$$
$$= \sum_{\nu=1}^{n} (x - \lambda_\nu)c_\nu = \sum_{\lambda_\nu < x} (x - \lambda_\nu)c_\nu.$$

Thus it appears that $(1/x)\int_0^x A(u)du$ is the typical mean (R, λ) of the first order of the series $\sum c_\nu$. Again

$$v(x) = \lambda_n s_n - \sum_{\nu=1}^{n} (\lambda_\nu - \lambda_{\nu-1})s_{\nu-1} = \sum_{\nu=1}^{n} \lambda_\nu c_\nu,\ \lambda_n \leqq x < \lambda_{n+1},$$

or

$$v(x) = \sum_{\lambda_\nu \leqq x} \lambda_\nu c_\nu.$$

Now assume

$$v_n = \sum_{\nu=1}^{n} \lambda_\nu c_\nu \geqq -K\lambda_n \qquad (n = 1, 2, 3, \cdots). \tag{17}$$

Then

$$\frac{v(x)}{x} = \frac{v_n}{x} \geqq -\frac{K\lambda_n}{x} \geqq -K,\ \lambda_n \leqq x < \lambda_{n+1},$$

and Theorem 1 yields

THEOREM 1′. *Conditions* (1), (2) *and* (17) *imply*

$$\sum_{\lambda_\nu < x} (x - \lambda_\nu)c_\nu = \int_0^x A(u)du \sim sx,\ x \to \infty. \tag{18}$$

6. Next we prove

Theorem 2. *Let*

$$\sum_{\lambda_\nu < x} (x - \lambda_\nu) c_\nu \sim sx, \quad x \to \infty, \tag{18}$$

and let

$$\sum_{\nu=2}^{n} | c_\nu |^p \lambda_\nu^p (\lambda_\nu - \lambda_{\nu-1})^{1-p} = O(\lambda_n) \ \text{as} \ n \to \infty, \quad p > 1. \tag{6'}$$

Then

$$s_n = A(\lambda_n) \to s \ \text{as} \ n \to \infty.$$

We start with the identity

$$\delta\lambda_n s_n = \int_0^{(1+\delta)\lambda_n} A(u)du - \int_0^{\lambda_n} A(u)du - \int_{\lambda_n}^{(1+\delta)\lambda_n} [A(u) - A(\lambda_n)]du,$$

or

$$\begin{aligned} s_n = & \frac{1}{(1+\delta)\lambda_n} \int_0^{(1+\delta)\lambda_n} A(u)du \cdot \frac{1+\delta}{\delta} - \frac{1}{\lambda_n} \int_0^{\lambda_n} A(u)du \cdot \frac{1}{\delta} \\ & - \frac{1}{\delta\lambda_n} \int_{\lambda_n}^{(1+\delta)\lambda_n} [A(u) - A(\lambda_n)]du, \quad \delta > 0. \end{aligned} \tag{19}$$

Here

$$A(u) - A(\lambda_n) = \begin{cases} \sum_{\nu=1}^{k} c_{n+\nu}, & \lambda_{n+k} \leqq u < \lambda_{n+k+1}, \quad k \geqq 1, \\ 0, & \lambda_n \leqq u < \lambda_{n+1}, \end{cases} \tag{20}$$

and

$$\left| \sum_{\nu=n+1}^{n+k} c_\nu \right| \leqq \sum_{\nu=n+1}^{n+k} | c_\nu | = \sum_{\nu=n+1}^{n+k} | c_\nu | \lambda_\nu (\lambda_\nu - \lambda_{\nu-1})^{-1/p'} \frac{(\lambda_\nu - \lambda_{\nu-1})^{1/p'}}{\lambda_\nu}.$$

By Hölder's inequality and by (6'),

$$\sum_{n+1}^{n+k} | c_\nu | = O[\lambda_{n+k}^{1/p} \lambda_{n+1}^{-1} (\lambda_{n+k} - \lambda_n)^{1/p'}], \quad p' = \frac{p}{p-1}.$$

Since $\lambda_{n+k} \leqq u \leqq (1+\delta)\lambda_n$, we have

$$\sum_{n+1}^{n+k} | c_\nu | = O[(1+\delta)^{1/p} \delta^{1/p'}] = O(\delta^{1/p'}),$$

and

$$A(u) - A(\lambda_n) = O(\delta^{1/p'}), \quad \lambda_n < u \leqq (1+\delta)\lambda_n, \quad n \to \infty.$$

For a fixed δ it follows from (18) and (19) that

$$\limsup_{n\to\infty} s_n \leqq s + O(\delta^{1/p'}),$$

and, on allowing $\delta \to 0$,

$$\limsup_{n\to\infty} s_n \leqq s.$$

A similar argument shows

$$\liminf_{n\to\infty} s_n \geqq s,$$

which completes the proof of Theorem 2.

Theorems 1′ and 2 immediately yield

THEOREM 3. *Condition* (2) *and* (6′) *imply that* $\sum_1^\infty c_\nu$ *converges to* s.

It is plain that (6′) implies the convergence of (1) for $t>0$. It remains only to observe that (6′) implies (17). Indeed

$$\begin{aligned}\sum_{\nu=1}^{n} \lambda_\nu |c_\nu| &= \sum_1^n \lambda_\nu |c_\nu| (\lambda_\nu - \lambda_{\nu-1})^{-1/p'}(\lambda_\nu - \lambda_{\nu-1})^{1/p'} \\ &\leqq \left(\sum_1^n |c_\nu|^p \lambda_\nu^p (\lambda_\nu - \lambda_{\nu-1})^{1-p}\right)^{1/p}\left(\sum_1^n (\lambda_\nu - \lambda_{\nu-1})\right)^{1/p'} \\ &= O(\lambda_n^{1/p+1/p'}) = O(\lambda_n).\end{aligned}$$

7. Another generalization of (6) is

$$(6'') \qquad \sum_1^n (|c_\nu| - c_\nu)^p \lambda_\nu^p (\lambda_\nu - \lambda_{\nu-1})^{1-p} = O(\lambda_n), \quad p > 1, \quad n \to \infty,$$

while

$$(21) \qquad \liminf_{n\to\infty} c_n \geqq 0.^*$$

It can be derived from (7), but we can prove it directly, by an argument used above.

First we have

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* If $\lambda_{n+1}/\lambda_n \to 1$, then (21) is a consequence of (6″) as will be shown later (cf. (21′)). For the case $\lambda_n = n$ cf. Szász [11].

$$-\sum_1^n \lambda_\nu c_\nu \leqq \sum_1^n \lambda_\nu(|c_\nu| - c_\nu) \leqq \left[\sum_1^n (|c_\nu| - c_\nu)^p \lambda_\nu^p (\lambda_\nu - \lambda_{\nu-1})^{1-p}\right]^{1/p} \lambda_n^{1/p'}$$

$$= O(\lambda_n).$$

Hence by Theorem 1′, (1), (2) and (6″) imply

$$\sum_{\lambda_\nu < x} (x - \lambda_\nu) c_\nu = \int_0^x A(u)du = A_1(x) \sim sx, \quad x \to \infty.$$

Next we have

$$-\sum_{n+1}^{n+k} c_\nu \leqq \sum_{n+1}^{n+k} (|c_\nu| - c_\nu)$$

$$\leqq \left[\sum_{n+1}^{n+k} (|c_\nu| - c_\nu)^p \lambda_\nu^p (\lambda_\nu - \lambda_{\nu-1})^{1-p}\right]^{1/p} \lambda_{n+1}^{-1} (\lambda_{n+k} - \lambda_n)^{1/p'},$$

hence by (20)

$$-\frac{1}{\delta\lambda_n} \int_{\lambda_n}^{(1+\delta)\lambda_n} [A(u) - A(\lambda_n)]du \leqq O(\lambda_{n+k}^{1/p} \lambda_{n+1}^{-1} \lambda_n^{1/p'} \delta^{1/p'}) = O(\delta^{1/p'}),$$

and so by (19)

$$\limsup_{n\to\infty} s_n \leqq s + O(\delta^{1/p'}).$$

On allowing here $\delta \to 0$ we get

$$\limsup_{n\to\infty} s_n \leqq s.$$

Furthermore, since

$$\text{(22)} \quad \frac{\delta}{1+\delta} \lambda_n s_n = \int_0^{\lambda_n} A(u)du - \int_0^{\lambda_n(1+\delta)^{-1}} A(u)du + \int_{\lambda_n(1+\delta)^{-1}}^{\lambda_n} [s_n - A(u)]du$$

and since $s_n = A(\lambda_n)$,

$$s_n - A(u) = \sum_{\nu=0}^{k} c_{n-\nu} \text{ for } \lambda_{n-k-1} \leqq u < \lambda_{n-k}, \quad k \geqq 0.$$

Now if $k \geqq 1$, and $\lambda_{n-k} > u \geqq \lambda_n(1+\delta)^{-1}$, then

$$-\sum_{\nu=0}^{k-1} c_{n-\nu} \leqq \sum_0^{k-1} (|c_{n-\nu}| - c_{n-\nu})$$

$$\leqq \left(\sum_{n-k+1}^{n} (|c_\nu| - c_\nu)^p \lambda_\nu^p (\lambda_\nu - \lambda_{\nu-1})^{1-p}\right)^{1/p} \lambda_{n-k}^{-1} (\lambda_n - \lambda_{n-k})^{1/p'},$$

and

$$-\sum_{0}^{k-1} c_{n-\nu} \leqq \frac{1+\delta}{\lambda_n}\lambda_n^{1/p'}\left(\frac{\delta}{1+\delta}\right)^{1/p'} O(\lambda_n^{1/p}) = O(\delta^{1/p'}).$$

Furthermore, by (6''),

$$(21') \qquad - c_{n-k} \leqq | c_{n-k} | - c_{n-k} = O\left(\left(\frac{\lambda_{n-k} - \lambda_{n-k-1}}{\lambda_{n-k}}\right)^{1/p'}\right).$$

Hence under either one of the assumptions $\lambda_{n+1}/\lambda_n \to 1$, or $\liminf_{n\to\infty} c_n \geqq 0$, it follows from (22) that

$$\liminf_{n\to\infty} s_n \geqq s - O(\delta^{1/p'}),$$

and on allowing $\delta \to 0$,

$$\liminf_{n\to\infty} s_n \geqq s.$$

This yields

THEOREM 4. *If* (1), (2) *and* (6'') *hold and if at least one of the additional conditions*

(a) $\dfrac{\lambda_{n+1}}{\lambda_n} \to 1,$ (b) $\liminf_{n\to\infty} c_n \geqq 0$

is satisfied, then $\sum_1^\infty c_\nu$ *converges to s.*

Notice that conditions (1), (2) and (6'') imply

$$\sum_{\lambda_\nu < x} (x - \lambda_\nu) c_\nu \sim s \cdot x, \text{ and } \limsup_{n\to\infty} s_n \leqq s,$$

but not the convergence in general. Even the more restrictive condition

$$\lambda_n c_n > - K(\lambda_n - \lambda_{n-1}) \qquad (n = 1, 2, 3, \cdots)$$

does not imply the convergence, as is shown by an example of Ananda-Rau [2].

8. We now shall state a theorem which includes as special cases not only the results of Landau and Neder but also condition (3) and even Theorem 3. On putting

$$\psi_n(\delta) = \underset{\lambda_{n+k} \leqq \lambda_n(1+\delta)}{\text{maximum}} | s_{n+k} - s_n |, \ \psi_n(\delta) = 0 \text{ if } \lambda_{n+k} > \lambda_n(1+\delta),$$

let us assume

$$\limsup_{n\to\infty} \psi_n(\delta) = \psi(\delta) \to 0 \text{ as } \delta \to 0.$$

This can be written in the form

$$\limsup_{n\to\infty} \underset{\lambda_n \leqq x \leqq (1+\delta)\lambda_n}{\text{maximum}} \left| A(x) - A(\lambda_n) \right| = \psi(\delta) \to 0, \quad \delta \to 0, \tag{23}$$

or

$$\left| A(x) - A(\lambda_n) \right| < \epsilon \text{ for } \lambda_n \leqq x \leqq (1+\delta)\lambda_n, \quad \delta = \delta(\epsilon). \tag{23'}$$

This condition is satisfied automatically if we have a series with gaps, that is, if for a constant $\theta > 1$

$$\lambda_{n+1} > \theta\lambda_n \qquad (n = 1, 2, 3, \cdots).$$

Assume the conditions of Theorem 1′, so that

$$\sum_{\lambda_\nu < x} (x - \lambda_\nu)c_\nu = A_1(x) \sim s \cdot x, \quad x \to \infty.$$

This and (23′) hold if we assume (2) and (6′).

Now using (19) and (23′) we get

$$\limsup_{n\to\infty} s_n \leqq s + \epsilon, \quad \liminf_{n\to\infty} s_n \geqq s - \epsilon.$$

Since ϵ is arbitrary, it follows that

$$\lim_{n\to\infty} s_n = \lim_{x\to\infty} A(x) = s.$$

Thus we obtain

THEOREM 5. *Conditions* (1), (2), (17) *and* (23) *imply*

$$\sum_{\nu=1}^{\infty} c_\nu = s.$$

9. Hardy and Littlewood have proved that from

$$\sum_{\nu=1}^{\infty} \left(\frac{\lambda_\nu}{\lambda_\nu - \lambda_{\nu-1}} \right)^\rho \left| c_\nu \right|^{\rho+1} < \infty, \quad \rho > 0,$$

and from (2) follows the convergence of $\sum c_\nu$. The following generalization is an easy consequence of Theorem 4.

THEOREM 6. *Conditions* (1), (2) *and*

$$\sum_{\nu=1}^{\infty} \left(\frac{\lambda_\nu}{\lambda_\nu - \lambda_{\nu-1}} \right)^\rho (\left| c_\nu \right| - c_\nu)^{\rho+1} < \infty, \quad \rho > 0,$$

imply $\sum_{\nu=1}^{\infty} c_\nu = s$.

For now we have

$$|c_\nu| - c_\nu = o\left(\left(\frac{\lambda_\nu - \lambda_{\nu-1}}{\lambda_\nu}\right)^{\rho/(\rho+1)}\right) = o(1),$$

hence condition (b) is satisfied. Moreover on setting

$$u_n = \sum_1^n \left(\frac{\lambda_\nu}{\lambda_\nu - \lambda_{\nu-1}}\right)^\rho (|c_\nu| - c_\nu)^{\rho+1},$$

$$\sum_1^n (|c_\nu| - c_\nu)^{\rho+1}\lambda_\nu^{\rho+1}(\lambda_\nu - \lambda_{\nu-1})^{-\rho} = u_n\lambda_n - \sum_1^{n-1} u_\nu(\lambda_{\nu+1} - \lambda_\nu) = o(\lambda_n);$$

(6′′) holds a posteriori and Theorem 6 is proved.

Finally we observe that condition

$$\sum_1^n a_\nu^p \lambda_\nu^p (\lambda_\nu - \lambda_{\nu-1})^{1-p} = O(\lambda_n), \tag{6a}$$

where a_ν stands for $|c_\nu|$ or for $|c_\nu| - c_\nu$, is equivalent to the following: there exists a constant $g > 1$ such that

$$\sum_{x<\lambda_\nu \leqq gx} a_\nu^p (\lambda_\nu - \lambda_{\nu-1})^{1-p} = O(x^{1-p}) \text{ as } x \to \infty. \tag{6b}$$

For from (6a) it follows that

$$\begin{aligned}\sum_{x<\lambda_\nu \leqq 2x} a_\nu^p (\lambda_\nu - \lambda_{\nu-1})^{1-p} &\leqq x^{-p} \sum_{x<\lambda_\nu \leqq 2x} a_\nu^p \lambda_\nu^p (\lambda_\nu - \lambda_{\nu-1})^{1-p} \\ &= O(x^{1-p}).\end{aligned}$$

Conversely on putting $x_\nu = \lambda_n g^{-\nu}$ $(\nu = 0, 1, 2, \cdots)$ we have

$$\begin{aligned}\sum_1^n a_\nu^p \lambda_\nu^p (\lambda_\nu - \lambda_{\nu-1})^{1-p} &= \sum_\nu \sum_{x_{\nu-1}<\lambda_k \leqq x_\nu} a_k^p \lambda_k^p (\lambda_k - \lambda_{k-1})^{1-p} \\ &\leqq \sum_\nu x_\nu^p \sum_{x_{\nu-1}<\lambda_k \leqq x_\nu} a_k^p (\lambda_k - \lambda_{k-1})^{1-p},\end{aligned}$$

and by (6b)

$$\sum_1^n a_\nu^p \lambda_\nu^p (\lambda_\nu - \lambda_{\nu-1})^{1-p} = O\left(\sum_\nu x_\nu\right) = O\left(\lambda_n \sum_0^\infty g^{-\nu}\right) = O(\lambda_n).^*$$

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* After this paper was completed and sent to the editors, the author learned of an interesting paper by G. Ricci, *Sui teoremi Tauberiani*, Annali di Matematica, (4), vol. 13 (1935), pp. 287–308, where bounds for oscillation of $A(x)$ are given, under the assumptions (2′) and $A(y) - A(x) > -K$ for $0 \leqq x \leqq y \leqq x(1+H)$.

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Brown University,
Providence, R. I., and
Massachusetts Institute of Technology,
Cambridge, Mass.